

# Fourier Analysis

Feb. 22, 2022

Review:

A convergence Thm: Let  $f$  be integrable on the circle.

Suppose  $f$  is differentiable at  $x_0$ . Then

$$S_N f(x_0) \rightarrow f(x_0) \quad \text{as } N \rightarrow \infty.$$

The above result is based on the Riemann-Lebesgue lemma.

Remark: Using the same proof, we can show that the above result holds under the weaker assumption that

$f$  is Lipschitz continuous at  $x_0$ , this is,

$\exists M > 0$  such that

$$(*) \quad |f(x) - f(x_0)| \leq M \cdot |x - x_0| \quad \text{for all } x \in [-\pi, \pi].$$

Corollary 1. Let  $f, g \in \mathbb{R}$ . Suppose that

$f$  and  $g$  coincide on an open interval  $I$  which contains  $x_0$ . Then

$$S_N f(x_0) - S_N g(x_0) \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Pf. Apply the convergence Thm to  $f - g$  at  $x_0$ . □

FACT: The above result implies that

the convergence of  $S_N f$  at  $x_0$  only depends on the behavior of  $f$  on a neighborhood of  $x_0$ .

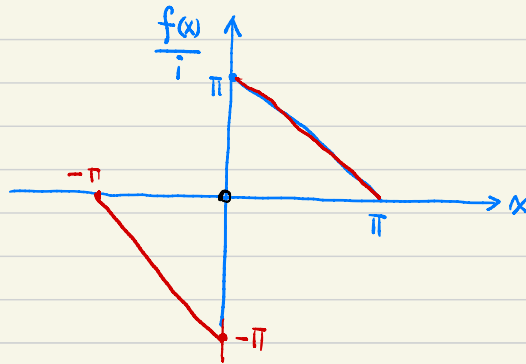
This fact is called "Riemann's localization principle".

### § 3.2

A continuous function on the circle whose Fourier series diverges at some point

Let us start from a special function

$$f(x) = \begin{cases} i(\pi - x) & \text{if } 0 \leq x \leq \pi \\ i(-\pi - x) & \text{if } -\pi \leq x < 0 \end{cases}$$



A direct calculation shows that

$$f(x) \sim \sum_{n \neq 0} \frac{1}{n} e^{inx} \quad \text{on } [-\pi, \pi]$$

For  $N \in \mathbb{N}$ , define

$$f_N(x) = \sum_{0 < |n| \leq N} \frac{1}{n} e^{inx} \quad (= S_N f(x))$$

$$\tilde{f}_N(x) = \sum_{n=-N}^{-1} \frac{1}{n} e^{inx}$$

Lem 2 : (1)  $\exists$  a constant  $M > 0$  such that

$$|f_N(x)| \leq M \quad \text{for all } N \in \mathbb{N} \text{ and } x \in [-\pi, \pi]$$

$$(2) \quad |\tilde{f}_N(0)| \geq \log N$$

Pf. We first prove (1).

Let us consider the Abel mean of  $f$ ,

$$A_r f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(x-y) f(y) dy,$$

$$\text{where } P_r(x) := \frac{1-r^2}{1-2r \cos x + r^2}, \quad 0 \leq r < 1.$$

Notice that for  $0 \leq r < 1$ ,

$$\begin{aligned} |A_r f(x)| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(x-y) |f(y)| dy \\ &\leq \frac{\|f\|_{\infty}}{2\pi} \int_{-\pi}^{\pi} P_r(x-y) dy \\ &= \|f\|_{\infty} \quad \left( \|f\|_{\infty} := \sup_{y \in [-\pi, \pi]} |f(y)| \right) \end{aligned}$$

However

$$\begin{aligned} &|f_N(x) - A_r f(x)| \\ &= \left| \sum_{0 < |n| \leq N} \frac{1}{n} e^{inx} - \sum_{|n| > 0} \frac{r^{|n|}}{n} e^{inx} \right| \\ &= \left| \sum_{0 < |n| \leq N} \frac{1-r^{|n|}}{n} e^{inx} - \sum_{|n| \geq N+1} \frac{r^{|n|}}{n} e^{inx} \right| \\ &\leq \left| \sum_{0 < |n| \leq N} \frac{1-r^{|n|}}{n} e^{inx} \right| + \left| \sum_{|n| \geq N+1} \frac{r^{|n|}}{n} e^{inx} \right| \end{aligned}$$

$$\leq \sum_{0 < |n| \leq N} \frac{1-r^{|n|}}{|n|} + \sum_{|n| \geq N+1} \frac{r^{|n|}}{|n|}$$

$$\leq 2 \cdot \sum_{n=1}^N \frac{1-r^n}{n} + 2 \cdot \sum_{n=N+1}^{\infty} \frac{r^n}{N}$$

Notice that  $\frac{1-r^n}{n} = \frac{(1-r)(1+r+r^2+\dots+r^{n-1})}{n}$

$$\leq 1-r.$$

Hence

$$|f_N(x) - A_r f(x)| \leq 2N(1-r) + 2 \sum_{n=N+1}^{\infty} \frac{r^n}{N}$$

$$\leq 2N(1-r) + 2 \cdot \frac{r^{N+1}}{N(1-r)}$$

$$\leq 2N(1-r) + \frac{2}{N(1-r)}$$

Let us take  $\tilde{r} = 1 - \frac{1}{N}$ , then  $N(1-\tilde{r}) = 1$

Then  $|f_N(x) - A_{\tilde{r}} f(x)| \leq 4$

It follows that

$$\begin{aligned} |f_N(x)| &\leq 4 + |A_T f(x)| \\ &\leq 4 + \|f\|_\infty \end{aligned}$$

This prove (1).

Next we prove (2), i.e.  $|\widehat{f}_N(0)| > \log N$

Notice that

$$\widehat{f}_N(0) = \sum_{n=-N}^{-1} \frac{1}{n} e^{in \cdot 0} = -\left[1 + \frac{1}{2} + \dots + \frac{1}{N}\right]$$

So

$$|\widehat{f}_N(0)| = 1 + \frac{1}{2} + \dots + \frac{1}{N}$$

$$\left( \text{using } \frac{1}{k} \geq \int_k^{k+1} \frac{1}{x} dx \right.$$

for all  $k > 0$ )

$$\geq \sum_{k=1}^N \int_k^{k+1} \frac{1}{x} dx$$

$$= \sum_{k=1}^N (\log(k+1) - \log k)$$

$$= \log(N+1) > \log N.$$



Next for  $N \in \mathbb{N}$ , define

$$P_N(x) = e^{2iNx} \cdot f_N(x)$$

$$= \sum_{\substack{n=-N \\ n \neq 0}}^N \frac{1}{n} e^{i(n+2N)x}$$

$$= \sum_{\substack{n=N \\ n \neq 2N}}^{3N} \frac{1}{n-2N} e^{inx}$$

$$\tilde{P}_N(x) = e^{2iNx} \cdot \tilde{f}_N(x) = \sum_{n=N}^{2N-1} \frac{1}{n-2N} e^{inx}$$

Clearly  $|P_N(x)| = |f_N(x)| \leq M,$

$$|\tilde{P}_N(x)| = |\tilde{f}_N(x)|, \quad |\tilde{P}_N(0)| > \log N.$$



Construct a sequence  $(N_k)_{k=1}^{\infty}$  of positive integers and a sequence  $(d_k)_{k=1}^{\infty}$  of positive numbers such that

$$(1) \quad N_{k+1} > 3 N_k$$

$$(2) \quad \sum_{k=1}^{\infty} d_k < \infty$$

$$(3) \quad d_k \log N_k \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

( For instance, we can take

$$N_k = 4^{4^k} \quad \text{and} \quad d_k = 3^{-k} )$$

Finally set

$$f(x) = \sum_{k=1}^{\infty} d_k P_{N_k}(x), \quad x \in [-\pi, \pi].$$

Since  $|P_{N_k}(x)| \leq M$  and  $\sum_{k=1}^{\infty} d_k M < \infty$ ,

by the Weierstrass M-Test, we have

$$\sum_{k=1}^l d_k P_{N_k}(x) \Rightarrow g \quad \text{on the circle} \\ \text{as } l \rightarrow \infty.$$

and  $g$  is continuous on the circle.

Next we claim that

$$S_N g(0) \not\rightarrow g(0) \quad \text{as } N \rightarrow \infty \quad (**)$$

Lem 3.

$$\hat{g}(n) = \begin{cases} \frac{d_k}{n - 2N_k} & \text{if } n \in [N_k, 3N_k] \\ & \text{and } n \neq 2N_k \\ 0 & \text{otherwise.} \end{cases}$$

Pf. Recall that for given  $N$ ,

$$P_N(x) = \sum_{\substack{N \leq n \leq 3N \\ n \neq 2N}} \frac{1}{n - 2N} e^{inx}$$

$$\text{Hence } \hat{P}_N(n) = \begin{cases} \frac{1}{n - 2N} & \text{if } n \in [N, 3N] \\ & n \neq 2N \\ 0 & \text{otherwise} \end{cases}$$

Now fix  $n \in \mathbb{Z}$ . Let  $\varepsilon > 0$ . Then  $\exists L \in \mathbb{N}$  such that

$$\left| g(x) - \sum_{j=1}^l \alpha_j P_{N_j}(x) \right| < \varepsilon \text{ if } l \geq L.$$

Observe that for  $h \in \mathcal{R}$ ,

$$\begin{aligned} |h(n)| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(x) e^{-inx}| dx \\ &\leq \|h\|_{\infty}. \end{aligned}$$

Hence

$$(***) \quad \left| \hat{g}(n) - \sum_{j=1}^l \alpha_j \widehat{P}_{N_j}(n) \right| < \varepsilon \text{ if } l \geq L$$

Now suppose that

$$n \notin \bigcup_{j=1}^{\infty} [N_j, 3N_j],$$

Then  $\alpha_j \widehat{P}_{N_j}(n) = 0$  for all  $j$  so

$$\sum_{j=1}^l \alpha_j \widehat{P}_{N_j}(n) = 0 \text{ for all } l.$$

So by (\*\*\*) ,

$$|\hat{g}(n)| \leq \varepsilon.$$

Since  $\varepsilon$  is arbitrarily given,  $\hat{g}(n) = 0$ .

Next assume  $n \in [N_k, 3N_k]$  for some  $k$ .

Then for any  $l \geq k$ ,

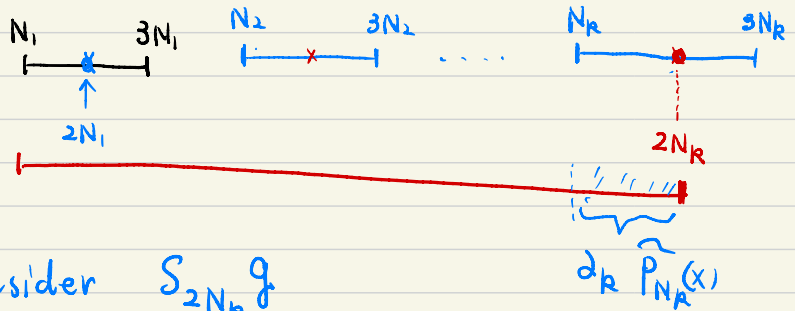
$$\begin{aligned} \sum_{j=1}^l d_j \hat{P}_{N_j}(n) &= d_R \hat{P}_{N_R}(n) \\ &= \begin{cases} d_R \cdot \frac{1}{n-2N_R} & \text{if } n \neq 2N_R \\ 0 & \text{if } n = 2N_R \end{cases} \end{aligned}$$

By (\*\*\*) , we see that

$$\hat{g}(n) = d_R \hat{P}_{N_R}(n)$$

This proves Lem 3. 

$$\bullet \quad g(x) = d_1 P_{N_1}(x) + d_2 P_{N_2}(x) + \dots + d_R P_{N_R}(x) + \dots$$



Let us consider  $S_{2N_R} g$

We notice that

$$S_{2N_R} g(x) = d_1 P_{N_1}(x) + \dots + d_{R-1} P_{N_{R-1}}(x) + d_R \cdot \widetilde{P}_{N_R}(x)$$

$$S_{2N_R} g(0) = \left( d_1 P_{N_1}(0) + \dots + d_{R-1} P_{N_{R-1}}(0) \right) + d_R \cdot \widetilde{P}_{N_R}(0)$$

$$= (\text{I}) + (\text{II})$$

$$|(\text{I})| \leq \sum_{i=1}^{R-1} d_i \cdot M < \sum_{i=1}^{\infty} d_i$$

but  $|(\text{II})| \geq d_R \log N_R \rightarrow +\infty$

Hence

$$\left| \int_{2N_R} g(z) \right| \rightarrow +\infty \neq g(0)$$

